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MAT109 Ordinary Differential Equation Project

Prepared by

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June 25, 2024

Project: Electric Circuit

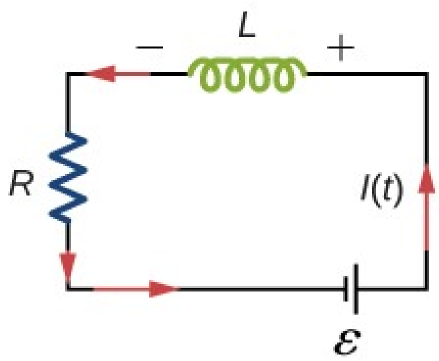

Figure 1: Simple electric system

If Q is the charge at time t in an electrical closed circuit with inductance L, resistance R, and capacitance C, then by Kirchhoff's Second Law (from Physics) the impressed voltage or called electromotive force E(t) is equal to the sum of the voltage drops in the rest of the circuit.
The relation between the charge Q and current I is

$$I = \frac{dQ}{dt} \tag{1}$$

According to the elementary laws of circuit, we know that:
The voltage drop across the resistor is RI.
The voltage drop across the capacitor is $\frac{Q}{C}$
The voltage drop across the inductor is $L\frac{dI}{dt}$.
Therefore, by the Kirchhoff's Law,

$$L\frac{dI}{dt} + RI + \frac{1}{c}Q = E(t) \tag{2}$$

Substitute for I from equation (1), we obtain a differential equation

$$LQ\text{"} + RQ' + \frac{1}{C}Q = E(t) \tag{3}$$

for the charge Q. In order to know the charge at any time, it is sufficient to know the charge on the capacitor and the current in the circuit at some initial time t_0.
Rearranging the equation, we can obtain a differential equation for the current I by differentiating equation (3) with respect to t, and then substituting for $\frac{dQ}{dt}$ from equation (1). The result that we obtained is

$$LI\text{"} + RI' + \frac{1}{C}I = E'(t), \tag{4}$$

With the initial conditions

$$I(t_0) = I_0, I'(t_0) = I'_0 \tag{5}$$

From equation (2), it follows that

$$I'_0 = \frac{E(t_0) - RI_0 - \frac{Q_0}{C}}{L}. \tag{6}$$

Hence, t_0 is also determined by the initial charge and current, which are physically measurable quantities.

We can divide equation (4) with L and we got

$$I" + \frac{R}{L}I' + \frac{1}{LC}I = \frac{E'(t)}{L} \tag{7}$$

Next, we can set $I(t) = y$, where $\begin{cases} \alpha = \frac{R}{L} > 0 \\ \omega = \frac{1}{LC} > 0 \end{cases}$

Then, we have

$$y" + \alpha y' + wy = \frac{E'(t)}{L}$$

First of all, we search the homogeneous solution of this equation, which is

y" $+\alpha y' + wy = 0$.

Our characteristic equation is:

$$r^2 + \alpha r + w = 0$$

Case 1: $\alpha^2 - 4\omega < 0$

$$\begin{cases} r_1 = -\frac{\alpha}{2} + \psi i \\ r_2 = -\frac{\alpha}{2} - \psi i \end{cases} \tag{8}$$

$$\psi = \frac{\sqrt{\alpha^2 - 4\omega}}{2} \tag{9}$$

The general solution is:

$$y_g(t) = e^{-\frac{\alpha}{2}t}\left(C_1 \cos(\psi t) + C_2 \sin(\psi t)\right) = A \cdot e^{-\frac{\alpha}{2}t} \sin(\psi t + \phi_0) \tag{10}$$

(A, ϕ_0 are constants such that $A > 0, \phi_0 \in (0, 2\pi)$)

Case 2: $\alpha^2 - 4\omega > 0$

$$r_3 = \frac{-\alpha + \sqrt{\alpha^2 - 4\omega}}{2} \tag{11}$$

$$r_4 = \frac{-\alpha - \sqrt{\alpha^2 - 4\omega}}{2} \tag{12}$$

The general solution is:

$$y_g(t) = C_1 e^{r_3 t} + C_2 e^{r_4 t} \tag{13}$$

Case 3: $\alpha^2 - 4\omega = 0$

$$r = r_5 = r_6 = -\frac{\alpha}{2} \tag{14}$$

The general solution is:

$$y_g(t) = C_1 e^{-\frac{\alpha}{2}t} + C_2 t e^{-\frac{\alpha}{2}t} \tag{15}$$

Let's consider the external power supply in the circuit as a stable power supply, which means E'(t) is just a constant. Then, we have E'(t) = 0

So, we obtain:

$$y\text{''} + \alpha y' + wy = 0$$

And the solution is the same of the homogeneous one.

Then, we consider a realistic problem of a RLC circuit which is commonly used in various applications such as filtering or signal tuning which can resonate at a frequency of 100 kHz. The component values for this circuit are as follows:

$$R = 10\,\Omega$$

$$L = 100\,\mu H = 1 \times 10^{-4}\,\mathrm{H}$$

$$C = 25\,\mathrm{pF} = 2.5 \times 10^{-11}\,\mathrm{F}$$

So:

$$\alpha = \frac{R}{L} = 10^5$$

$$\omega = \frac{1}{LC} = 4 \times 10^{14}$$

Then,

$$\alpha^2 - 4\omega = (10^5)^2 - 4 \times (4 \times 10^{14}) = 10^{10} - 16 \times 10^{14} < 0$$

So, it's the case 1 and here we have:

$$\begin{cases} y(t) = Ae^{-\frac{\alpha}{2}t} \sin(\psi t + \phi_0) \\ y'(t) = Ae^{-\frac{\alpha}{2}t} \left(\psi \cos(\psi t + \phi_0) - \frac{\alpha}{2} \sin(\psi t + \phi_0)\right) \end{cases}$$

Let's set the current I_0 at time $t_0 = 0$ to 0.5 A and the derivative of the current with respect to time: $\frac{dI}{dt} = I_0$ at time $t_0 = 0$ to 0.06 A/s.

So,

$$\begin{cases} y(0) = I_0 = 0.5 \\ y'(0) = I_0' = 0.06 \end{cases}$$

Substituting the initial values that we have into the equation y(t) and y'(t), we got

$$y(0) = A \sin(\phi_0) = I_0 \tag{16}$$

$$y'(0) = A \left(\psi \cos(\phi_0) - \frac{\alpha}{2} \sin(\phi_0)\right) \tag{17}$$

We can acquire A from equation (16):

$$A = \frac{I_0}{sin(\phi_0)}$$

Then, we can substitute the A that we got into the equation (17), and the result is:

$$\phi_0 = \cot^{-1}\left(\frac{I_0'}{2 \times 10^7 I_0} + 2.5 \times 10^{-3}\right)$$

By plugging in the value of I_0 and I_0':

$A \approx 0.5$

$\phi_0 \approx 1.57$ rad

Finally, by substituting A and ϕ_0 back to the y(t), we have:

$$I(t) = y = 0.5e^{(-5 \times 10^4)t} \times \sin(2 \times 10^7 t + 1.57) \tag{18}$$