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Mathematical Analysis II Report

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INTRODUCTION

In this report, we focus on the concept of distance between sets and the related theorems. Specifically, we begin by proving that the distance between two disjoint sets—a compact set A and a closed set C—is strictly positive. Building on this result, we proceed to establish the Strong Separating Hyperplane Theorem. Finally, we demonstrate an application of the Strong Separating Hyperplane Theorem by using it to prove Farkas' Lemma.

DISTANCE BETWEEN SETS

In this section, we give the definition of the distance between sets and present a proof of our first theorem.

Definition 1 (Distance Between Sets)**.** Let A and B be two subsets of $\mathbb{R}^n$. The distance between A and B is defined as

$$d(A, B) = \inf \{d(a, b) \mid a \in A,\ b \in B\}.$$

Here, $d(a, b) = \|a - b\|$ denotes the distance between the points in the Euclidean space $\mathbb{R}^n$.

Lemma 1. Given a subset A of $\mathbb{R}^n$, define the function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = d(x, A)$. Then f is a continuous function.

Proof. Given u and v in $\mathbb{R}^n$, if $a \in A$, by the definition of Distance Between Sets, we have

$$d(u, A) \leqslant \|u - a\| \leqslant \|v - a\| + \|u - v\|.$$

This shows that

$$\|v - a\| \geqslant d(u, A) - \|u - v\|$$

Since this inequality holds for every $a \in A$, the infimum over $a \in A$ on the right-hand side gives

$$d(v, A) \geqslant d(u, A) - \|u - v\|$$

Therefore,

$$f(u) - f(v) \leqslant \|u - v\|$$

Switching u and v we have

$$f(v) - f(u) \leqslant \|u - v\|$$

Above two inequalities proves that

$$|f(u) - f(v)| \leqslant \|u - v\|$$

This shows that f is a Lipschitz function with Lipschitz constant 1, which implies that it is continuous.

Theorem 1. Let A and C be disjoint subsets of $\mathbb{R}^n$. If A is compact and C is closed, then the distance between A and C is positive.

Proof. Since A and C are disjoint subsets of $\mathbb{R}^n$,

$$A \cap C = \varnothing.$$

So for $\forall a \in A, a \in \mathbb{R}^n \setminus C$.
Since C is closed, $\mathbb{R}^n \setminus C$ is open, thus for $\forall a \in A, \exists r > 0$

$$B(a, r) \in \mathbb{R}^n \setminus C.$$

This imply that

$$\|a - c\| \geqslant r > 0, \forall c \in C.$$

Define the function $f : A \to \mathbb{R}$ by $f(a) = d(a, C)$. By $Lemma 1$, f is continuous, since A is compact, by the Extreme Value Theorem, f has a minimum value, means that $\exists a_0 \in A$, such that

$$f(a_0) = \min_{a \in A} f(a) > 0.$$

Hence,

$$d(A, C) = \underset{a \in A, c \in C}{inf} \|a - c\| = \min_{a \in A} f(a) = f(a_0) > 0.$$

STRONG SEPARATING HYPERPLANE THEOREM

The Strong Separating Hyperplane Theorem is a fundamental result in convex analysis. It asserts that if two nonempty, closed convex sets in $\mathbb{R}^n$ are disjoint , then there exists a hyperplane that strictly separates them—that is, each set lies entirely in a different open half-space defined by the hyperplane.This theorem has been widely applied in optimization theory, support vector machines, and functional analysis. In this section, we provide the statement and a proof of the theorem.

Definition 2 (Convex Sets)**.** Let $S \subseteq \mathbb{R}^n$. We say that S is *convex* if for any two points $u, v \in S$, the line segment connecting u and v lies entirely in S. Equivalently, S is *convex* if for all $u, v \in S$ and all $t \in [0,1]$, the point

$$(1-t)u + tv \in S.$$

Theorem 2 (Strong Separating Hyperplane Theorem)**.** Let C and D be disjoint nonempty convex subsets of $\mathbb{R}^n$. Suppose C is compact and D is closed. Then there exists

$$a \in \mathbb{R}^n, a \neq 0, \gamma \in \mathbb{R}$$

such that

$$a^\top x < \gamma, \forall x \in C$$
$$a^\top y > \gamma, \forall y \in D.$$

Proof. Let C and D be disjoint nonempty convex subsets of $\mathbb{R}^n$, C is compact and D is closed. By $Theorem 1$,

$$dist\left(C, D\right) > 0$$

and there exist $x_0 \in C, y_0 \in D$ such that

$$\left\|x_0 - y_0\right\| = inf\left\{\left\|x - y\right\|, x \in C, y \in D\right\} = dist\left(C, D\right) > 0$$

Let $a = y_0 - x_0 \neq 0$, then

$$\left\|a\right\| = \left\|x_0 - y_0\right\| > 0$$

Let $f : C \to \mathbb{R}, f\left(x\right) = \left\|x - y_0\right\|^2$, by the definition of Distance Between Sets,

$$f\left(x\right) \geqslant \left(dist\left(C, D\right)\right)^2, \forall x \in C.$$

Let $t \in [0,1], z_1 : [0,1] \to \mathbb{R}^n, z_1\left(t\right) = x_0 + t\left(x - x_0\right)$ and
$\varphi_1 : [0,1] \to \mathbb{R}, \varphi_1\left(t\right) = f\left(x_0 + t\left(x - x_0\right)\right)$, we have

$$\begin{aligned}
\varphi_1(t) &= \|x_0 + t(x - x_0) - y_0\|^2 = \|z_1(t) - y_0\|^2 \\
&= (z_1(t) - y_0)^\top (z_1(t) - y_0)\,, \\
\varphi_1'(t) &= 2\,(z_1(t) - y_0)^\top z_1'(t) \\
&= 2\,(z_1(t) - y_0)^\top (x - x_0), \\
\varphi_1'(0) &= 2(x_0 - y_0)^\top (x - x_0).
\end{aligned}$$

Since $f\left(x\right) \geqslant \left(dist\left(C, D\right)\right)^2, \forall x \in C$,

$$\varphi_1{}'\left(0\right) = 2\left(x_0 - y_0\right)^\top\left(x - x_0\right) \geqslant 0.$$

This imply that

$$\begin{aligned}
2\left(x_0 - y_0\right)^\top\left(x - x_0\right) = -2a^\top\left(x - x_0\right) \geqslant 0, \\
a^\top\left(x - x_0\right) \leqslant 0,
\end{aligned}$$

$$a^\top x \leqslant a^\top x_0, \forall x \in C.$$

Following similar arguments, we also arrive at

$$a^\top y \geqslant a^\top y_0.$$

Let $\gamma = \frac{1}{2}\left(a^\top x_0 + a^\top y_0\right)$, then

$$\begin{aligned}
a^\top y_0 - \gamma = a\gamma - a^\top x_0 &= \frac{1}{2}\left(a^\top y_0 - a^\top x_0\right) \\
&= \frac{1}{2} a^\top\left(a\right) \\
&= \frac{1}{2}\left\|a\right\|^2 > 0.
\end{aligned}$$

Therefore,

$$a^\top x \leqslant a^\top x_0 < \gamma < a^\top y_0 \leqslant a^\top y, \forall x \in C, y \in D,$$

which can be concluded that

$$a^\top x < \gamma < a^\top y, \forall x \in C, y \in D.$$

This proves the Strong Separating Hyperplane Theorem.

APPLICATION OF THE STRONG SEPARATING HYPERPLANE THEOREM

In this section, we delve into the concrete application of the Strong Separating Hyperplane Theorem, specifically its role in analyzing the existence of solutions to linear systems. Interestingly, this line of reasoning ultimately leads us to a well-known result in convex analysis: Farkas' Lemma.

Definition 3 (Nonnegative Vector)**.** A vector $x \in \mathbb{R}^m$ is *nonnegative*, denoted as

$$x \geqslant 0$$

if every component of x is nonnegative, that is $x_i \geq 0$ for all $i = 1, 2, \dots, m$.

Example 1. Consider a system of linear equations of the form $Ax = b$, where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. A fundamental question in linear analysis is whether there exists a vector $x \in \mathbb{R}^n$such that the equation holds.

Solution 1. Let $K := \{Ax \mid x \geqslant 0\}$ be a subset of $\mathbb{R}^m$, where $x \in \mathbb{R}^m$ is *nonnegative*, $B := \{b \mid b \in \mathbb{R}^m\}$ be a subset of $\mathbb{R}^m$ which is a singleton set.

Thus, the problem of whether a system of linear equations of the form $Ax = b$ has solution become the question of whether K and B are disjoint.
We now consider two distinct cases:

- **Case 1:** $b \in K$. In this case, by the definition of K, there exists a nonnegative vector $x \in \mathbb{R}^m$ such that $Ax = b$. Therefore, the system has a solution.
- **Case 2:** $b \notin K$. In this case, since b is the only element in B, so the sets K and B are disjoint. For any point u and v in K, by the definition of K, there exists $x_1, x_2 \in \mathbb{R}^n$, such that

$$\begin{aligned} u &= Ax_1 \\ v &= Ax_2, \end{aligned}$$

therefore, for $t \in [0, 1]$,

$$\begin{aligned} u + t\,(v - u) &= Ax_1 + t\,(Ax_2 - Ax_1) \\ &= Ax_1 + At\,(x_2 - x_1) \\ &= A\,(x_1 + t\,(x_2 - x_1)) \\ &= A\,((1 - t)\,x_1 + tx_2) \\ &= Ax_3, \end{aligned}$$

and for $t \in [0,1]$, $x_3 = (1-t)\,x_1 + tx_2 \in \mathbb{R}^m$ is nonnegative, thus

$$u + t\,(v-u) \in K, \forall t \in [0,1]\,.$$

This imply that K is convex.
The set $B := \{b \mid b \in \mathbb{R}^m\}$ is convex, since any singleton set is convex.
Since K is closed and B is closed and bounded—hence compact—and both K and B are nonempty, the Strong Separating Hyperplane Theorem can be applied. By the Strong Separating Hyperplane Theorem, there exists

$$y \in \mathbb{R}^n, y \neq 0, \gamma \in \mathbb{R}$$

such that

$$\begin{aligned} y^\top x < \gamma, \forall x \in B \\ y^\top x > \gamma, \forall x \in K. \end{aligned}$$

Thus we have

$$y^\top k > \gamma > y^\top b, \forall k \in K.$$

Take $k_0 = 0$, then $y^\top k_0 = 0$, since the inequality holds for all $k \in K$, we obtain

$$0 > \gamma > y^\top b.$$

Since $y^\top b = b^\top y$, we conclude that

$$b^\top y < 0.$$

Now we focus on $A^\top y$, we clain that $A^\top y \geqslant 0$. We have proved that for all $k \in K$, $y^\top k > \gamma > y^\top b$ and $b^\top y < 0$. Let $b^\top y = m < 0$. Then consider the contradiction, we assume that the i-th component of $A^\top y$ is negative, namely, $\left(A^\top y\right)_i = n < 0$ then consider the sequence $\{x_j\}_{j=0}^{\infty} \in K \subset \mathbb{R}^n$, where all the components of x_j are 0 except the i-th component—is j.

Since

$$\begin{aligned} y^\top k = y^\top Ax &= \left(A^\top y\right)^\top x \\ &= x^\top \left(A^\top y\right). \end{aligned}$$

For $x_i \in \{x_j\}_{j=0}^{\infty}$, since all the components of x_i are 0 except the i-th component—is j,

$$\begin{aligned} y^\top k &= {x_i}^\top \left(A^\top y\right) \\ &= j \cdot \left(A^\top y\right)_i \\ &= j \cdot n. \end{aligned}$$

Since both n and m are fixed negative numbers, there exists $j_1 \in \mathbb{Z}$, such that $\forall j > j_1$,

$$y^\top k = j \cdot n < m = b^\top y,$$

which is contradicts to $y^\top k > b^\top$.
This proved that $A^\top y \geqslant 0$.

To summarize, we have shown that for a system of linear equations of the form $Ax = b$, where $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. There is exactly one of the following two assertions is true:

(i) There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$;

(ii) There exists $y \in \mathbb{R}^m$ such that $A^\top y \geq 0$ and $b^\top y < 0$.

This statement is also one of the many equivalent formulations of $Farkas' Lemma$ that appear in the literature. The version presented here is equivalent to the one given by Gale, Kuhn, and Tucker.

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