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MAT201 Mathematical Analysis I
Project 2

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INTRODUCTION

In this report, we will discuss several issues related to integration. Specifically, we will explore how Darboux sums can be used to explain the definition of Riemann integrability for a bounded function and provide an equivalent definition of Riemann integrability in terms of Riemann sums. Additionally, this report will prove the Archimedes-Riemann theorem and present a concrete example illustrating its application. Finally, we will prove two versions of the fundamental theorem of calculus and also provide two explicit examples demonstrating their applications.

RIEMANN INTEGRABILITY FOR A BOUNDED FUNCTION AND AN EQUIVALENT DEFINITION OF RIEMANN INTEGRABILITY

In this section, we will first explain how Darboux sums can be used to explain the definition of Riemann integrability for a bounded function.

Definition 1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[a, b]$. For each subinterval $[x_{i-1}, x_i]$, define

$$m_i = \inf_{x\in[x_{i-1},x_i]} f(x), \quad M_i = \sup_{x\in[x_{i-1},x_i]} f(x).$$

The Darboux lower sum $L(f, P)$ and the Darboux upper sum $U(f, P)$ are defined as:

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Definition 2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The lower integral of f is defined as:

$$\underline{\int_a^b} f = sup \left\{ L\left(f, P\right) | P \ is \ a \ partition \ of \ \ [a, b] \right\},$$

where $L(f, P)$ is the Darboux lower sums over all partitions P of $[a, b]$.
Similarly, the upper integral of f is defined as:

$$\overline{\int_a^b} f = inf \left\{ U\left(f, P\right) | P \ is \ a \ partition \ of \ \ [a, b] \right\},$$

where $U(f, P)$ is the Darboux upper sum over all partitions P of $[a, b]$.

Based on the definitions of the Darboux lower sum, Darboux upper sum, lower integral and upper integral, we can define Riemann Integrability as follows:

Definition 3. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. The function f is said to be *Riemann integrable* on $[a,b]$ if and only if its lower integral equals its upper integral:

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

In this case, the common value is called the Riemann integral of f over $[a,b]$, denoted by:

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

Now we are going to provide an equivalent definition of Riemann integrability, which will involve the use of Riemann sums.

Definition 4. Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[a,b]$, where $a = x_0 < x_1 < \cdots < x_n = b$. For each subinterval $[x_{i-1}, x_i]$, choose a point $c_i \in [x_{i-1}, x_i]$. The *Riemann sum* associated with the partition P and the choice of points $A = \{c_i\}_{i=1}^n$ is defined as:

$$R\left(f, P, A\right) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

Definition 5. Let $f : [a,b] \to \mathbb{R}$ be a bounded function, $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[a,b]$, for each subinterval $[x_{i-1}, x_i]$, choose a point $c_i \in [x_{i-1}, x_i]$. There is a number I such that, for $\forall\varepsilon > 0, \exists\delta > 0$, if $|P| < \delta$, then the Riemann sum associated with the partition P and the choice of points $A = \{c_i\}_{i=1}^n$:

$$\left|R\left(f, P, A\right) - I\right| < \varepsilon.$$

In that case, the function f is said to be *Riemann integrable* on $[a,b]$, and

$$I = \int_a^b f.$$

Archimedes-Riemann theorem and example of the application of the Archimedes-Riemann theorem

In this section, we will give a proof of the Archimedes-Riemann theorem and provide an example of its application.

Lemma 6. $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for any $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that:

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, from *Definition 3* we have:

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

Given $\varepsilon > 0$, by definitions of lower and upper integrals, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$L(f, P_1) > \int_a^b f - \frac{\varepsilon}{2},$$
$$U(f, P_2) < \int_a^b f + \frac{\varepsilon}{2}.$$

Let P be a common refinement of P_1 and P_2. Then

$$L(f, P_1) \leqslant L(f, P) \leqslant U(f, P) \leqslant U(f, P_2).$$

Therefore,

$$U(f, P) - L(f, P) \leqslant U(f, P_2) - L(f, P_1) < \varepsilon.$$

Conversely, if for any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$, then for every positive integer n, take $\varepsilon = \frac{1}{n}$ we have:

$$U(f, P) - L(f, P) < \varepsilon = \frac{1}{n}.$$

By the definition of lower integral and upper integral,

$$\underline{\int_a^b} f - \overline{\int_a^b} f < \varepsilon = \frac{1}{n}.$$

Since $U(f, P) \geqslant L(f, P)$,

$$0 \leqslant \underline{\int_a^b} f - \overline{\int_a^b} f < \varepsilon = \frac{1}{n}.$$

Take $n \to \infty$, by *Squeeze Theorem*,

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

This shows that f is Riemann integrable.

Theorem 7 (Archimedes-Riemann Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if there is a sequence P_n of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0.$$

In this case, the Riemann integral of f over $[a, b]$ can be computed by

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n).$$

Proof. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, by *Lemma 6*, for every positive integer n, there is a partition P_n of $[a, b]$ such that

$$0 \leqslant U(f, P_n) - L(f, P_n) < \frac{1}{n}.$$

By *Squeeze Theorem*,

$$\lim_{n\to\infty} U\left(f, P_n\right) - L\left(f, P_n\right) = 0.$$

Conversely, if there is a sequence P_n of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} U\left(f, P_n\right) - L\left(f, P_n\right) = 0,$$

by the definition of limit of sequences, we have for $\forall\varepsilon > 0$, $\exists N \in \mathbb{Z}^+$ such that for all $n \geqslant N$,

$$\left|U\left(f, P_n\right) - L\left(f, P_n\right)\right| < \varepsilon.$$

Since $U(f, P) \geqslant L(f, P)$,

$$U\left(f, P_n\right) - L\left(f, P_n\right) < \varepsilon.$$

By *Lemma 6*, f is Riemann integrable. Therefore,

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f,$$

$$0 \leqslant U\left(f, P_n\right) - \int_a^b f \leqslant U\left(f, P_n\right) - L\left(f, P_n\right).$$

Take $n \to \infty$, by *Squeeze Theorem*,

$$\lim_{n\to\infty} U\left(f, P_n\right) = \int_a^b f.$$

Since $\lim_{n\to\infty}(U\left(f, P_n\right) - L\left(f, P_n\right)) = 0$,

$$\int_a^b f = \lim_{n\to\infty} U\left(f, P_n\right) = \lim_{n\to\infty} L\left(f, P_n\right).$$

The following is an example of a practical application of the Archimedes-Riemann Theorem.

Example 1. Let $f : [0, \pi] \to \mathbb{R}$ be the function $f(x) = \cos(x)$, Show that f is Riemann integrable and find the integral $\int_0^\pi f(x)\,dx$.

Proof. Let P_n be the regular partition of $[0, \pi]$ into n intervals. Since for $x \in (0, \pi)$,

$$f'(x) = -\sin(x) < 0,$$

so f is strictly decreasing on $[0, \pi]$.

For each subinterval $\left[\frac{i\pi}{n}, \frac{(i+1)\pi}{n}\right]$, $i \in \mathbb{Z}, 0 \leqslant i \leqslant n-1$,

$$m_i = f\left(\frac{(i+1)\pi}{n}\right) = \cos\left(\frac{(i+1)\pi}{n}\right),$$

$$M_i = f\left(\frac{i\pi}{n}\right) = \cos\left(\frac{i\pi}{n}\right).$$

Therefore,

$$U\left(f, P_n\right) = \sum_{i=0}^{n-1} \frac{\pi}{n} \cdot M_i = \sum_{i=0}^{n-1} \frac{\pi}{n} \cdot \cos\left(\frac{i\pi}{n}\right),$$

$$L\left(f, P_n\right) = \sum_{i=0}^{n-1} \frac{\pi}{n} \cdot m_i = \sum_{i=0}^{n-1} \frac{\pi}{n} \cdot \cos\left(\frac{(i+1)\pi}{n}\right),$$

$$\begin{aligned}
U\left(f, P_n\right) - L\left(f, P_n\right) &= \sum_{i=0}^{n-1} \frac{\pi}{n} \cdot \left[\cos\left(\frac{i\pi}{n}\right) - \cos\left(\frac{(i+1)\pi}{n}\right)\right] \\
&= \frac{\pi}{n} \sum_{i=0}^{n-1} \left[\cos\left(\frac{i\pi}{n}\right) - \cos\left(\frac{(i+1)\pi}{n}\right)\right] \\
&= \frac{\pi}{n} \sum_{i=0}^{n-1} \left[\cos\left(\frac{i\pi}{n}\right) - \cos\left(\frac{i\pi}{n}\right)\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{i\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)\right] \\
&= \frac{\pi}{n} \sum_{i=0}^{n-1} \left\{\cos\left(\frac{i\pi}{n}\right)\left[1 - \cos\left(\frac{\pi}{n}\right)\right] + \sin\left(\frac{i\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)\right\}
\end{aligned}$$

Take $n \to \infty$, then

$$\begin{aligned}
\lim_{n\to\infty} U\left(f, P_n\right) - L\left(f, P_n\right) &= \lim_{n\to\infty} \frac{\pi}{n} \sum_{i=0}^{n-1} \left\{\cos\left(\frac{i\pi}{n}\right)\left[1 - \cos\left(\frac{\pi}{n}\right)\right] + \sin\left(\frac{i\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)\right\} \\
&= \lim_{n\to\infty} \frac{\pi}{n} \sum_{i=0}^{n-1} \left\{\cos\left(\frac{i\pi}{n}\right)\left[1 - \cos\left(\frac{\pi}{n}\right)\right]\right\} + \lim_{n\to\infty} \frac{\pi}{n} \sum_{i=0}^{n-1} \left[\sin\left(\frac{i\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)\right]
\end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{\pi}{n} = 0,$$

$$\lim_{n\to\infty} \sin\left(\frac{i\pi}{n}\right) = \lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = 0,$$

$$\lim_{n\to\infty} \left[1 - \cos\left(\frac{\pi}{n}\right)\right] = 0,$$

$$\lim_{n\to\infty} \cos\left(\frac{i\pi}{n}\right) = 1,$$

so

$$\lim_{n\to\infty} \frac{\pi}{n} \sum_{i=0}^{n-1} \left\{\cos\left(\frac{i\pi}{n}\right)\left[1 - \cos\left(\frac{\pi}{n}\right)\right]\right\} = \lim_{n\to\infty} \frac{\pi}{n} \sum_{i=0}^{n-1} \left[\sin\left(\frac{i\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)\right] = 0,$$

$$\lim_{n\to\infty} U\left(f, P_n\right) - L\left(f, P_n\right) = 0.$$

By *Archimedes-Riemann Theorem*, f is Riemann integrable, and

$$\begin{aligned}\int_0^\pi f(x)\,dx &= \lim_{n\to\infty} U(f, P_n) \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\pi}{n} \cdot \cos\left(\frac{i\pi}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{\pi}{n} \\ &= \lim_{n\to\infty} \frac{\pi}{n} \cdot n \\ &= \pi\end{aligned}$$

FUNDAMENTAL THEOREM OF CALCULUS AND EXAMPLES OF THE APPLICATIONS OF THE FUNDAMENTAL THEOREM OF CALCULUS

In this section, we will give the proofs two versions of the Fundamental Theorem of Calculus and provide examples demonstrating the practical applications of each theorem.

Theorem 8 (Fundamental Theorem of Calculus I)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function that is Riemann integrable, and let $F : [a, b] \to \mathbb{R}$ be the function defined by

$$F(x) = \int_a^x f(u)\,du.$$

If x_0 is a point in (a, b), and f is continuous at x_0, then F is differentiable at x_0 and

$$F\prime(x_0) = f(x_0).$$

Proof. Given $\varepsilon > 0$, since f is continuous at x_0, $\exists\ \delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and for $\forall x \in (x_0 - \delta, x_0 + \delta)$,

$$|f(x) - f(x_0)| < \frac{\varepsilon}{2}.$$

For $\forall h \in (-\delta, \delta)$,

$$F(x_0 + h) - F(x_0) - f(x_0)h = \int_a^{x_0+h} f(u)\,du - \int_a^{x_0} f(u)\,du - f(x_0)h$$

$$\begin{aligned} &= \int_{x_0}^{x_0+h} f(u)\,du - f(x_0)h \\ &= \int_{x_0}^{x_0+h} [f(u) - f(x_0)]\,du.\end{aligned}$$

Since $x_0 + h \in (x_0 - \delta, x_0 + \delta)$,

$$|F(x_0 + h) - F(x_0) - f(x_0)h| = \left| \int_{x_0}^{x_0+h} [f(u) - f(x_0)]\, du \right| \leqslant \frac{\varepsilon}{2} |h|.$$

Take $h \neq 0$, then

$$\left| \frac{F(x_0 + h)}{h} - \frac{F(x_0)}{h} - f(x_0) \right| \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

By the definition of limit of a function,

$$\lim_{h \to 0} \left(\frac{F(x_0 + h)}{h} - \frac{F(x_0)}{h} \right) = f(x_0),$$

$$\lim_{h \to 0} \frac{F(x_0 + h) - F(x_0)}{h} = f(x_0).$$

By the definition of derivative, F is differentiable at x_0 and

$$F\prime(x_0) = f(x_0).$$

Example 2. Evaluate the following derivative:

$$\frac{d}{dx} \int_{x^2}^{x^3} e^{\sin(u)} du.$$

Solution 1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{\sin(x)}$, since f is continuous on $\mathbb{R}$, it is Riemann integrable over any closed and bounded intervals. Let $F(x) = \int_0^x e^{\sin(u)} du$. By *Fundamental Theorem of Calculus I*,

$$F\prime(x) = f(x) = e^{\sin(x)}.$$

Therefore,

$$\frac{d}{dx} \int_{x^2}^{x^3} e^{\sin(u)} du = \frac{d}{dx} \left(\int_0^{x^3} e^{\sin(u)} du - \int_0^{x^2} e^{\sin(u)} du \right)$$

$$\begin{aligned} &= \frac{d}{dx} \left(F\left(x^3\right) - F\left(x^2\right) \right) \\ &= 3x^2 F\prime\left(x^3\right) - 2xF\prime\left(x^2\right) \\ &= 3x^2 e^{\sin\left(x^3\right)} - 2xe^{\sin\left(x^2\right)} \end{aligned}$$

We have already proved Fundamental Theorem of Calculus I and provided an example of its application. Next, we will prove Fundamental Theorem of Calculus II and likewise present an example of its application.

Lemma 9. Assume that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$, differentiable on (a, b), and for all $x \in (a, b)$,

$$f\prime(x) = g\prime(x).$$

Then there is a constant C such that for all $x \in [a, b]$,

$$f(x) = g(x) + C.$$

Proof. Define the function $h : [a, b] \to \mathbb{R}$ by

$$h(x) = f(x) - g(x).$$

Then the function h is continuous on $[a, b]$, differentiable on (a, b), and $h\prime(x) = 0$ for $\forall x \in (a, b)$. Take any $x \in (a, b]$, by*Mean Value Theorem*, there $\exists c \in (a, x)$ such that

$$\frac{h(x) - h(a)}{x - a} = h\prime(c) = 0.$$

This proves that $h(x) = h(a)$. Therefore, the function h is a constant. Namely, there is a constant C so that $h(x) = C$ for all $x \in [a, b]$. Therefore, for all $x \in [a, b]$,

$$f(x) = g(x) + C.$$

Lemma 10. Let $f : [a, b] \to \mathbb{R}$ be a bounded function that is Riemann integrable, and let $F : [a, b] \to \mathbb{R}$ be the function defined by

$$F(x) = \int_a^x f(u)du.$$

Then $F : [a, b] \to \mathbb{R}$ is a Lipschitz function, and hence it is continuous.

Proof. Since $f : [a, b] \to \mathbb{R}$ is bounded, there is a positive constant M such that for all $x \in [a, b]$,

$$|f(x)| \leqslant M.$$

For any $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, we have

$$F(x_1) - F(x_2) = \int_{x_1}^{x_2} f(u)du.$$

Therefore,

$$\begin{aligned} |F(x_1) - F(x_2)| = \left| \int_{x_1}^{x_2} f(u)du \right| &\leqslant \int_{x_1}^{x_2} |f(u)|du \\ &\leqslant \int_{x_1}^{x_2} Mdu = M(x_2 - x_1) = M|x_2 - x_1| \end{aligned}$$

This proves that $F : [a, b] \to \mathbb{R}$ is a Lipschitz function wth Lipschitz constant M, and hence it is continuous.

Theorem 11 (Fundamental Theorem of Calculus II)**.** Let $F : [a, b] \to \mathbb{R}$ be a continuous function, and let $f : [a, b] \to \mathbb{R}$ be a bounded function that is continuous on (a, b). If for all $x \in (a, b)$,

$$F\prime(x) = f(x).$$

Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Since $f : [a, b] \to \mathbb{R}$ is a bounded function that is continuous on (a, b), it is integrable over any subinterval of $[a, b]$. Let $G : [a, b]ßR$ be the function defined by

$$G(x) = \int_a^x f(u)du.$$

By *Lemma 10*, G is continuous on $[a, b]$. By *Fundamental Theorem of Calculus I*, for all $x \in (a, b)$,

$$G\prime(x) = f(x).$$

Hence, $F : [a, b] \to \mathbb{R}$ and $G : [a, b] \to \mathbb{R}$ are continuous functions satisfying for all $x \in (a, b)$,

$$G\prime(x) = F\prime(x).$$

By *Lemma 9*, there is a constant C such that

$$G(x) = F(x) + C.$$

Since $G(a) = 0$, we find that $C = -F(a)$. Therefore,

$$G(x) = F(x) - F(a),$$

$$\int_a^b f(x)dx = G(b) = F(b) - F(a).$$

Example 3. A car is traveling on a straight road, and its velocity $v(m/s)$ at time $t \in [0, 10]$ (s) satisfies $v = f(t) = t^2 + 1$. Calculate the distance traveled by the car over the time interval $[0, 10]$.

Solution 2. It is easy to see that the distance traveled by the car over the time interval $[0, 10]$ is the integral of f over the interval $[0, 10]$.
Namely, the distance traveled by the car over the time interval $[0, 10]$ D:

$$D = \int_0^{10} f(t)dt.$$

Since f is a polynomial, f is continuous on $(0, 10)$. We have $F(t) = \frac{1}{3}t^3 + t$ that satisfying

$$F\prime(x) = t^2 + 1 = f(t).$$

Since F is a polynomial, F is continuous on $[0, 10]$. By *Fundamental Theorem of Calculus II*,

$$\begin{aligned} D = \int_0^{10} f(t)dt &= F(10) - F(0) \\ &= \frac{1}{3}(10)^3 + 10 - \left[\frac{1}{3}(0)^3 + 0\right] \\ &= \frac{1030}{3} \end{aligned}$$

So the distance traveled by the car over the time interval $[0, 10]$ is $\frac{1030}{3}\ m$.