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# MAT201 Mathematical Analysis I
# Project 1

prepared by

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## Problem 1

(a) Prove the theorem which says that a bounded monotonic sequence must be convergent.

(b) Prove the theorem which says that a bounded sequence must have a convergent subsequence.

(c) Give an example of a sequence $\{a_n\}$ that is defined recursively, which you can use part (a) to conclude its convergence. Find the limit of the sequence.

(d) Recall that the number $e$ is defined as the limit of the sequence $\{a_n\}$, where

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Prove that this sequence is indeed convergent.

(e) Let $\{b_n\}$ be the sequence that is defined recursively by $b_1 = 2$, and for $n \geq 2$,

$$b_n = b_{n-1} + \frac{1}{n!}.$$

Show that $\lim\limits_{n\to\infty} b_n = e$.

***Solution:***

(a) *Proof.* ①

Given$\{a_n\}$ be a increasing sequence which is bounded above, so

$$c = sup\,\{a_n\}$$

exists.

Given $\varepsilon > 0, c - \varepsilon < c$, so $c - \varepsilon$ is not an upper bound of $\{a_n\}$, so

$$\exists N \in \mathbb{Z}^+, a_N > c - \varepsilon.$$

Since $\{a_n\}$ is increasing, so

$$\forall n \geqslant N, a_n \geqslant a_N > c - \varepsilon.$$

Since c is an upper bound of $\{a_n\}$, so

$$\forall n \in \mathbb{Z}^+, a_n \leqslant c < c + \varepsilon.$$

So

$$\forall n \geqslant N, |a_n - c| < \varepsilon.$$

This proves that $\{a_n\}$ converges to c.

②
Given$\{a_n\}$ be a decreasing sequence which is bounded below, so

$$c = inf\,\{a_n\}$$

exists.

Given $\varepsilon > 0, c + \varepsilon > c$, so $c + \varepsilon$ is not an lower bound of $\{a_n\}$, so

$$\exists N \in \mathbb{Z}^+, a_N < c + \varepsilon.$$

Since $\{a_n\}$ is decreasing, so

$$\forall n \geqslant N, a_n \leqslant a_N < c + \varepsilon.$$

So

$$\forall n \geqslant N, |a_n - c| < \varepsilon.$$

This proves that $\{a_n\}$ converges to c.

By combining results from ① and ②, we arrive at the conclusion that bounded monotonic sequence must be convergent.

(b) *Proof.* Given$\{a_n\}$ be a bounded sequence, let $J$ be its set of peak indices.

$$J = \{n_1, n_2, \cdots\} \subset \mathbb{Z}^+$$

$$n_1 < n_2 < \cdots .$$

① Case of $J$ is infinite.
The element in $J$ can be listd down as

$$n_1, n_2, \cdots$$

$$n_1 < n_2 < \cdots .$$

Then

$$a_{n_1} \geqslant a_{n_2} \geqslant a_{n_3} \geqslant \cdots .$$

So $\{a_{n_k}\}$ is a decreasing subsequence.
So $\{a_n\}$ has a monontone subsequence.

② Case of $J$ is finite or empty.
If $J = \phi$, let $n_1 = 1$

if $J \neq \phi$, let $n_1 = maxJ + 1$.
In any case, $n_1$ is not a peak index of $\{a_n\}$, so $\exists n_2 > n_1$ such that

$$a_{n_2} > a_{n_1}.$$

$n_2$ is not a peak index of $\{a_n\}$, so $\exists n_3 > n_2$ such that

$$a_{n_3} > a_{n_2}.$$

Arguing in this way, we find

$$n_1 < n_2 < \cdots ,$$

and

$$a_{n_1} < a_{n_2} < a_{n_3} < \cdots .$$

So $\{a_{n_k}\}$ is a increasing subsequence.
So $\{a_n\}$ has a monontone subsequence.

By combining results from ① and ②, we arrive at the conclusion that $\{a_n\}$ has a monontone subsequence $\{a_{n_k}\}$. Since $\{a_n\}$ is a bounded sequence, so $\{a_{n_k}\}$ is a bounded monontonic subsequence of $\{a_n\}$. By (a), $\{a_{n_k}\}$ is convergent, this proves that a bounded sequence must have a convergent subsequence.

(c)

**Definition 1.** Define the sequence$\{a_n\}$inductively by $a_1 = 1$ and for all $n \geqslant 1$,

$$a_{n+1} = \frac{3a_n + 3}{a_n + 3}.$$

Then,

$$a_1 = 1, a_2 = \frac{3 \times 1 + 3}{1 + 3} = \frac{3}{2} > a_1$$
$$a_3 = \frac{3 \times \frac{3}{2} + 3}{\frac{3}{2} + 3} = \frac{5}{3} > a_2$$
$$\vdots$$

For $n \geqslant 2$,

$$\begin{aligned} a_{n+1} - a_n &= \frac{3a_n + 3}{a_n + 3} - \frac{3a_{n-1} + 3}{a_{n-1} + 3} \\ &= \frac{(3a_n + 3)(a_{n-1} + 3) - (3a_{n-1} + 3)(a_n + 3)}{(a_{n-1} + 3)(a_n + 3)} \\ &= \frac{6a_n - 6a_{n-1}}{(a_{n-1} + 3)(a_n + 3)} \\ &= \frac{6(a_n - a_{n-1})}{(a_{n-1} + 3)(a_n + 3)} \end{aligned}$$

We proved by induction that $a_{n+1} > a_n > 0$.
Since $a_2 > a_1 > 0$, the statement is true when $n = 1$.
Assume that $a_n > a_{n-1} > 0$ when $n \geqslant 2$, then

$$a_n + 3 > 0, a_{n-1} + 3 > 0$$

so

$$a_{n+1} - a_n > 0$$

so

$$a_{n+1} > a_n.$$

By pricinple of mathematical induction,

$$\forall n \geqslant 1, a_{n+1} > a_n.$$

So $\{a_n\}$ is an increasing sequence. Since

$$a_{n+1} = \frac{3a_n + 3}{a_n + 3} < \frac{3a_n + 3}{a_n + 1} = 3$$

so $\{a_n\}$ is bounded above by 3. By monontone convergence theorem, $\{a_n\}$ is convergent. Let $a = \lim_{n\to\infty} a_n = sup\{a_n\} \geqslant 1$, then

$$a = \lim_{n\to\infty} a_{n+1}.$$

Since $a_{n+1} = \frac{3a_n+3}{a_n+3} < \frac{3a_n+3}{a_n+1} = 3$, so

$$\lim_{n\to\infty} a_{n+1} = \frac{\lim_{n\to\infty}(3a_n + 3)}{\lim_{n\to\infty}(a_n + 3)}$$

$$a = \frac{3a + 3}{a + 3}$$

$$a^2 + 3a = 3a + 3$$

$$a = \sqrt{3}.$$

So the limit of $\{a_n\}$ is $\sqrt{3}$.

(d) *Proof.* Since $a_n = \left(1 + \frac{1}{n}\right)^n$, apparently,$a_n > 0$, so

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \left(\frac{n+2}{n+1}\right)^{n+1}\left(\frac{n}{n+1}\right)^n \\ &= \left(\frac{n+2}{n+1}\right)\left(\frac{n^2+2n}{n^2+2n+1}\right)^n.\end{aligned}$$

We have

$$\left(\frac{n^2+2n}{n^2+2n+1}\right)^n = \left(1-\frac{1}{n^2+2n+1}\right)^n$$

let $c = \frac{1}{n^2+2n+1} \in (0,1)$, then

$$\begin{aligned}\left(1-\frac{1}{n^2+2n+1}\right)^n = (1-c)^n \geqslant 1 - nc &= 1-\frac{n}{n^2+2n+1}\\ &= \frac{n^2+n+1}{n^2+2n+1}\end{aligned}$$

then

$$\begin{aligned}\frac{a_{n+1}}{a_n} = \left(\frac{n+2}{n+1}\right)\left(1-\frac{1}{n^2+2n+1}\right)^n \geqslant &\left(\frac{n+2}{n+1}\right)\left(\frac{n^2+n+1}{n^2+2n+1}\right)\\ &= \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1}\\ &= 1+\frac{1}{(n+1)^3} > 1.\end{aligned}$$

Since $a_n > 0$, $a_{n+1} > a_n$, $\{a_n\}$ is increasing,

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n,$$

so

$$a_n = \left(1+\frac{1}{n}\right)^n = 1 + \binom{n}{1}\left(\frac{1}{n}\right) + \cdots + \binom{n}{k}\left(\frac{1}{n}\right)^k + \cdots + \binom{n}{n}\left(\frac{1}{n}\right)^n.$$

We have

$$\begin{aligned}\binom{n}{k}\left(\frac{1}{n}\right)^k &= \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k}\\ &= \frac{1}{k!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-k}{n}\right)\\ &\leqslant \frac{1}{k!} \leqslant \frac{1}{2^{k-1}}.\end{aligned}$$

So

$$\begin{aligned} a_n &\leqslant 1 + \frac{1}{2^0} + \frac{1}{2^1} + \cdots + \frac{1}{2^{n-1}} \\ &= 2 + 1 - \frac{1}{2^{n-1}} \\ &\leqslant 3. \end{aligned}$$

This proves that $\{a_n\}$ is bounded above.

By monontone convergence theorem, $\{a_n\}$ is convergent.

(e) *Proof.*

$$\begin{aligned} a_n &= \left(1+\frac{1}{n}\right)^n \\ &= 1 + \binom{n}{1}\left(\frac{1}{n}\right) + \cdots + \binom{n}{k}\left(\frac{1}{n}\right)^k + \cdots + \binom{n}{n}\left(\frac{1}{n}\right)^n \\ &= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}\cdot\frac{1}{n^k} + \cdots + \frac{n(n-1)\cdots(n-n+1)}{1\cdot 2\cdots n}\cdot\frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \\ &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \cdots + \frac{1}{n!} = b_n \end{aligned}$$

So we have $a_n < b_n$.

$$\begin{aligned} a_n &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \\ &> 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \end{aligned}$$

Take $n \to \infty$ to both side, by the defination of $e$, we have

$$\begin{aligned} \lim_{n\to\infty} a_n &= e \\ &\geqslant 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} \\ &= b_n \\ &= \lim_{n\to\infty} 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \end{aligned}$$

So we have $e \geqslant b_n$.
So we have $a_n < b_n \leqslant e$.
Take $n \to \infty$, we have

$$e < b_n \leqslant e.$$

By Squeeze Theorem, $\{b_n\}$ converges to e.
This proves that $\lim_{n\to\infty} b_n = e$.

**Problem 2**

(a) State and prove the intermediate value theorem.

(b) Give an explicit example which is an application of the intermediate value theorem.

(c) Use the intermediate value theorem to show that if $p(x)$ is a polynomial of degree $n$ and $n$ is an odd integer, then $p(x)$ has a real root.

(d) Given an example of a monic polynomial $p(x)$ of degree 3 with integer coefficients, which has exactly one real root $x_0$ that is not a rational number. Plot the graph of the function $p(x)$. Use bisection method to find the root $x_0$.

***Solution:***

(a)

**Theorem 2.** Given that $f : [a, b] \to \mathbb{R}$ is a continuous function. For any real number $w$ that is between $f(a)$ and $f(b)$, there exists a point $c$ in $[a, b]$ such that $f(c) = w$.

*Proof.* Consider a function $f : [a, b] \to \mathbb{R}$, assume that

$$f(a) < f(b), and\ f(a) < w < f(b).$$

Let

$$a_1 = a, b_1 = b, m_1 = \frac{a_1 + b_1}{2},$$

if $f(m_1) < w$, let

$$a_2 = m_1, b_2 = b_1,$$

if $f(m_1) \geqslant w$, let

$$a_2 = a_1, b_2 = m_1.$$

Then

$$f(a_2) < w \leqslant f(b_2)$$
$$b_2 - b_1 = \frac{b_1 - a_1}{2}$$
$$a_1 \leqslant a_2 < b_2 \leqslant b_1$$

Assume that we have found $[a_1, b_1], [a_2, b_2], \cdots [a_{n-1}, b_{n-1}]$ such that

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{n-1} < b_{n-1} \leqslant b_{n-2} \leqslant \cdots \leqslant b_1.$$

$$b_k - a_k = \frac{b_{k-1} - a_{k-1}}{2}$$
$$f(a_k) < w \leqslant f(b_k).$$

Let
$$m_{n-1}=\frac{a_{n-1}+b_{n-1}}{2},$$
$$a_{n-1}<m_{n-1}<b_{n-1},$$
$$m_{n-1}-a_{n-1}=b_{n-1}-m_{n-1}=\frac{b_{n-1}-a_{n-1}}{2},$$
if $f\left(m_{n-1}\right)<w$, let
$$a_n=m_{n-1}, b_n=b_{n-1},$$
if $f\left(m_{n-1}\right)\geqslant w$, let
$$a_n=a_{n-1}, b_n=m_{n-1}.$$
In any case we have
$$f\left(a_n\right)<w\leqslant f\left(b_n\right)$$
and
$$b_n-b_n=\frac{b_{n-1}-a_{n-1}}{2}.$$
Hence, this produces the sequences $\{a_n\}$ and $\{b_n\}$
$$a_1\leqslant a_2\leqslant\cdots\leqslant a_{n-1}<b_{n-1}\leqslant b_{n-2}\leqslant\cdots\leqslant b_1.$$
that $\{a_n\}$ is an increasing sequence that is bounded above by $b_1$,
$\{b_n\}$ is a decreasing sequence that is bonded below by $a_1$.
By monontone convergence theorem,
$$c_1=\lim_{n\to\infty}a_n, c_2=\lim_{n\to\infty}b_n$$
exists.
Since $b_n-b_n=\frac{b_{n-1}-a_{n-1}}{2}$,
$$\lim_{n\to\infty}\left(b_n-a_n\right)=\frac{1}{2}\lim_{n\to\infty}\left(b_{n-1}-a_{n-1}\right)$$
$$c_2-c_1=\frac{1}{2}\left(c_2-c_1\right)c_2-c_1=0, c_2=c_1.$$
Let $c=c_2=c_1$, then
$$\lim_{n\to\infty}a_n=c\lim_{n\to\infty}b_n=c.$$
Since $f:[a,b]\to\mathbb{R}$ is continuous,
$$\lim_{n\to\infty}f\left(a_n\right)=f\left(c\right)\ \lim_{n\to\infty}f\left(b_n\right)=f\left(c\right).$$
Since $f\left(a_n\right)\leqslant w$,
$$f\left(c\right)=\lim_{n\to\infty}f\left(a_n\right)\leqslant w.$$
Since $f\left(b_n\right)\geqslant w$,
$$f\left(c\right)=\lim_{n\to\infty}f\left(a_n\right)\geqslant w.$$

So

$$f(c) \leqslant w \leqslant f(c)\, f(c) = w.$$

This proves the intermediate value theorem.

(b)

**Example 1.** Show that the equation $x^6 + 6x + 1 = 0$ has a real root.

*Proof.* Define that $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^6 + 6x + 1.$$

Since $f$ is a polynomial, it is continuous, and

$$f(0) = 1 > 0 f(-1) = (-1)^6 + 6(-1) + 1 = -4 < 0 f(-1) < 0 < f(0).$$

By intermediate value theorem, $\exists c \in (-1, 1)$ such that

$$f(c) = 0.$$

This proves that the equation $x^6 + 6x + 1 = 0$ has a real root.

(c) *Proof.* Given a polynomial $p(x)$ of degree $n$ and $n$ is an odd integer, then

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \ \ (a_n \neq 0).$$

Let $q(x) = a_k x^k - a_{k-1} x^{k-1}, k \in (1, 2, \cdots, n) \subset \mathbb{Z}$.
Since $q$ is a polynomial, $q(x)$ is continuous, then

$$\lim_{x\to\infty} q(x) = \lim_{x\to\infty} a_k x^k - a_{k-1} x^{k-1} = \lim_{x\to\infty} x^{k-1} (a_k x - a_{k-1}).$$

Since $a_k$ are constants,

$$\lim_{x\to\infty} a_k x - a_{k-1} = \infty, (a_k > 0),$$

$$\lim_{x\to\infty} a_k x - a_{k-1} = -\infty, (a_k < 0),$$

and since $\lim_{x\to\infty} x^{k-1} = \infty$,

$$\lim_{x\to\infty} q(x) = \infty = \lim_{x\to\infty} a_k x^k, (a_k > 0),$$

$$\lim_{x\to\infty} q(x) = -\infty = \lim_{x\to\infty} a_k x^k, (a_k < 0).$$

Hence,

$$\lim_{x\to\infty} p\left(x\right) = \lim_{x\to\infty} a_n x^n,$$

if $a_n > 0$,

$$\lim_{x\to\infty} p\left(x\right) = \infty > 0,$$

$$\lim_{x\to-\infty} p\left(x\right) = -\infty < 0,$$

if $a_n < 0$,

$$lim_{x\to\infty} p\left(x\right) = -\infty < 0,$$

$$\lim_{x\to-\infty} p\left(x\right) = \infty > 0.$$

By intermediate value theorem, $\exists c \in \mathbb{R}$ such that

$$p\left(c\right) = 0.$$

This proves that if $p(x)$ is a polynomial of degree $n$ and $n$ is an odd integer, then $p(x)$ has a real root.

(d)

**Example 2.** Given that $p\left(x\right) = x^3 - x + 1$.
Since

$$p\left(0\right) = 1 > 0,$$

$$p\left(-2\right) = \left(-2\right)^3 - \left(-2\right) + 1 = -5 < 0,$$

$$p\left(\sqrt{\frac{1}{3}}\right) = \left(\sqrt{\frac{1}{3}}\right)^3 - \left(\sqrt{\frac{1}{3}}\right) + 1 = 1 - \sqrt{\frac{4}{27}} > 0,$$

by intermediate value theorem, $\exists c_1 \in (-2, 0)$ such that

$$p\left(c_1\right) = 0.$$

The first derivative of $p(x)$ is

$$p\prime\left(x\right) = 3x^2 - 1$$

,

$$p\prime\left(x\right) \geqslant 0, x \in \left[-\infty, -\sqrt{\frac{1}{3}}\right], \left[\sqrt{\frac{1}{3}}, \infty\right]$$

$$p\prime\left(x\right) < 0, x \in \left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right).$$

So $p(x)$ is increasing on $\left[-\infty, -\sqrt{\frac{1}{3}}\right], \left[\sqrt{\frac{1}{3}}, \infty\right]$, decreasing on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$.
So the minimum value of $p(x)$ on $\left(-\sqrt{\frac{1}{3}}, \infty\right)$ is $p(-\sqrt{\frac{1}{3}})$.
Since

$$p\left(\sqrt{\frac{1}{3}}\right) = \left(\sqrt{\frac{1}{3}}\right)^3 - \left(\sqrt{\frac{1}{3}}\right) + 1 = 1 - \sqrt{\frac{4}{27}} > 0,$$

so

$$p(x) > 0, x \in \left(-\sqrt{\frac{1}{3}}, \infty\right).$$

By intermediate value theorem, $\nexists c_2 \in \left(-\sqrt{\frac{1}{3}}, \infty\right)$ such that

$$p\left(c_2\right) = 0.$$

Since $p(x)$ is increasing on $\left[-\infty, -\sqrt{\frac{1}{3}}\right]$,$p\left(-2\right) = \left(-2\right)^3 - \left(-2\right) + 1 = -5 < 0$, so

$$p(x) < 0, x \in \left(-\infty, 2\right).$$

By intermediate value theorem, $\nexists c_3 \in (-\infty, 2)$ such that

$$p\left(c_2\right) = 0.$$

Hence, $p(x)$ has exactly one real root $x_0$ on $(-2, 0)$.

**Lemma 3.** In algebra, the rational root theorem (or rational root test, rational zero theorem, rational zero test or p/q theorem) states a constraint on rational solutions of a polynomial equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0.$$

with integer coefficients $a_i \in \mathbb{Z}, and\ a_0, a_n \neq 0$. Solutions of the equation are also called roots or zeros of the polynomial on the left side. The theorem states that each rational solution $x = \frac{p}{q}$, written in lowest terms so that $p$ and $q$ are relatively prime, satisfies:

$$p\ is\ an\ integer\ factor\ of\ the\ cons\tan t\ term\ a_0, and$$

$$q\ is\ an\ integer\ factor\ of\ the\ leading\ coefficient\ a_n.$$

By Lemma 3, the possible rational roots can only be $1$ or $-1$.

$$p(1) = 1^3 - 1 + 1 = 1 \neq 0.$$

$$p(-1) = (-1)^3 - (-1) + 1 = 1 \neq 0.$$

Since neither of these possible rational roots satisfies the equation, it indicates that the polynomial has no rational roots.
Therefore, the root $x_0$ is not a rational number.

The following figure is $p(x)$.

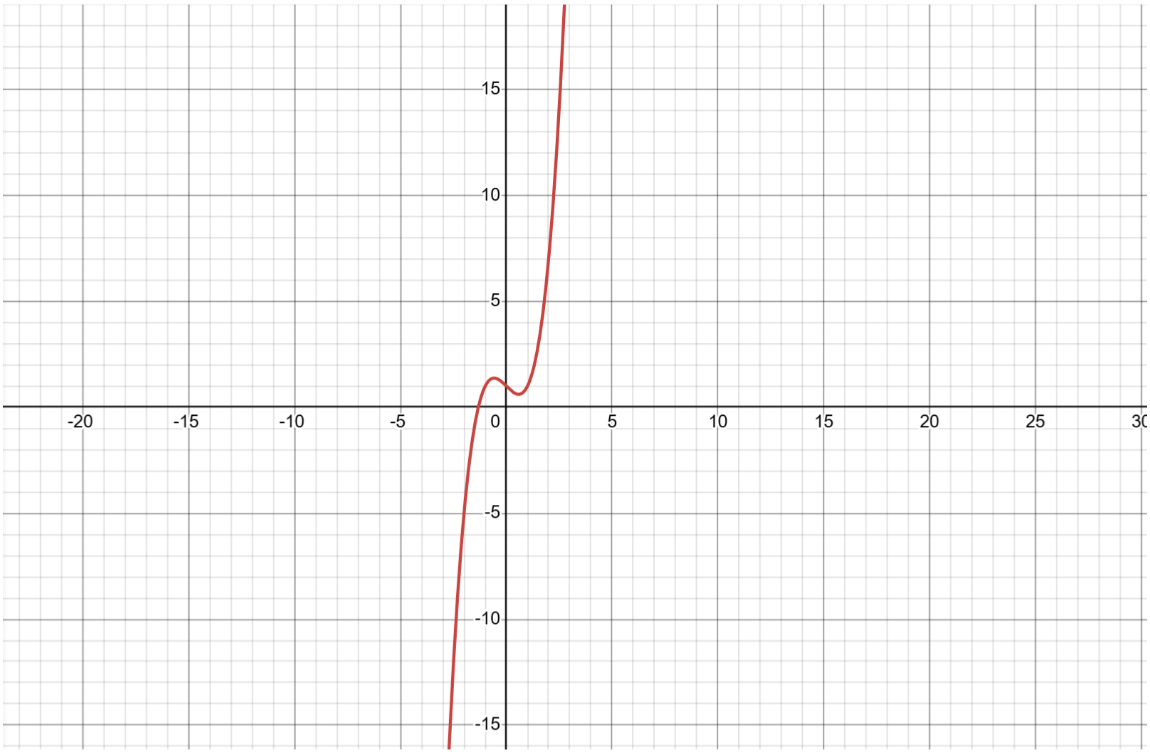


Figure 1: figure of p(x)

**Problem 3**

(a) State and prove the Lagrange's mean value theorem.

(b) Give an explicit example which uses intermediate value theorem and mean value theorem to conclude the number of solutions of an equation.

(c) Explain how the Lagrange's mean value theorem can be used to determine the monotonicity of a function.

(d) Give an example to illustrate part (c).

***Solution:***

(a)

**Theorem 4.** Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies the following conditions:

- $f : [a, b] \to \mathbb{R}$ is continuous.
- $f : [a, b] \to \mathbb{R}$ is differentiable.

Then there is a point $x_0$ in $(a, b)$ such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

*Proof.*

**Lemma 5.** Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies the following conditions:

- $f : [a, b] \to \mathbb{R}$ is continuous.
- $f : [a, b] \to \mathbb{R}$ is differentiable.
- $f(a) = f(b)$.

Then there is a point $x_0$ in $(a, b)$ such that

$$f'(x_0) = 0.$$

Let $g\left(x\right) = f\left(x\right) - mx$, where $m$ is chosen such that

$$g\left(a\right) = g\left(b\right)$$

$$f\left(a\right) - mx = f\left(b\right) - mx$$

$$m\left(b - a\right) = f\left(b\right) - f\left(a\right)$$

$$m = \frac{f\left(b\right) - f\left(a\right)}{b - a}$$

$$g\left(x\right) = f\left(x\right) - mx.$$

Since $g\left(x\right) = f\left(x\right) - mx$,

$$g : [a, b] \to \mathbb{R}\ is\ continous.$$

$$g : (a, b) \to \mathbb{R}\ is\ differentiable.$$

$$g\prime\left(x\right) = f\prime\left(x\right) - m.$$

Since $g\left(a\right) = g\left(b\right)$, by lemma 5,

$$\exists c \in (a, b)$$

such that

$$g\prime\left(c\right) = 0.$$

Therefore

$$f\prime\left(c\right) = m = \frac{f\left(b\right) - f\left(a\right)}{b - a}.$$

This proves that the Lagrange's mean value theorem.

(b)

**Example 3.** Given the equation $x^6 + 6x + 1 = 0$, determine the exact number of real roots of this equation.

*Proof.* Let $f : \mathbb{R} \to \mathbb{R}$ be a function

$$f\left(x\right) = x^6 + 6x + 1.$$

Since it is a polynomial, it is differentiable and

$$f\prime\left(x\right) = 6x^5 + 6.$$

Let $f\prime\left(x\right) = 0$, then

$$x^5 + 1 = 0$$

$$x = -1.$$

If $x_1 < x_2$,and $f(x_1) = f(x_2) = 0$, by lemma 5,

$$\exists c_1 \in (x_1, x_2)$$

such that

$$f\prime\left(c_1\right) = 0.$$

So if $x_1 < x_2 < x_3$,and $f(x_1) = f(x_2) = f(x_3) = 0$, then

$$\exists c_1 \in (x_1, x_2)$$

$$\exists c_2 \in (x_2, x_3)$$

such that

$$f\prime(c_1) = f\prime(c_2) = 0.$$

Notice that $c_1 \neq c_2$.
Since if $f\prime(x) = 0$, then $x = -1$.
Therefore, $f(x) = 0$ can have at most 2 different solutions.
Since $f(x) = x^6 + 6x + 1$, so

$$f(0) = 1 > 0$$

$$f(-1) = (-1)^6 + 6(-1) + 1 = -4 < 0$$

$$f(-2) = (-2)^6 + 6(-2) + 1 = 53 > 0.$$

By intermediate value theorem,

$$\exists x_1 \in (-1, 0)$$

such that

$$f(x_1) = 0$$

$$\exists x_2 \in (-2, -1)$$

such that

$$f(x_2) = 0.$$

Therefore, $f(x) = 0$ can have at least 2 different solutions.
We arrive at the conclusion that $f(x) = 0$ have exactly 2 different solutions.
Therefore, the equation $x^6 + 6x + 1 = 0$ have exactly 2 different roots.

(c) Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies the following conditions:

- $f : [a, b] \to \mathbb{R}$ is continuous.
- $f : [a, b] \to \mathbb{R}$ is differentiable.

Then $f$ applies to Lagrange's mean value theorem.
So $\forall (x_1, x_2) \in [a, b]$,

$$\exists x_0 \in (x_1, x_2)$$

such that

$$f\prime(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $x_2 > x_1$,
if $f\prime(x_0) < 0$, then

$$f(x_2) < f(x_1)$$

then $f : [a, b] \to \mathbb{R}$ is strictly decreasing,
if $f\prime(x_0) > 0$, then

$$f(x_2) > f(x_1)$$

then $f : [a, b] \to \mathbb{R}$ is strictly increasing.
The above shows that Lagrange's mean value theorem can be used to derive the role of the first-order derivative in determining the monotonicity of a function.

(d)

**Example 4.** Show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is strictly increasing.

*Proof.* Since $f$ is a polynomial, $f$ is continuous and differentiable on $\mathbb{R}$.

Then $f$ applies to Lagrange's mean value theorem.
So $\forall\,(x_1, x_2) \in \mathbb{R}$,

$$\exists x_0 \in (x_1, x_2)$$

such that

$$f\prime\,(x_0) = \frac{f\,(x_2) - f\,(x_1)}{x_2 - x_1}.$$

The first-order derivative of $f$ :

$$f\prime\,(x) = 3x^2$$

, so

$$f\prime\,(x) > 0, \forall x \in \mathbb{R},$$

$$f\prime\,(x_0) = \frac{f\,(x_2) - f\,(x_1)}{x_2 - x_1} > 0.$$

Since $x_2 > x_1$,

$$f\,(x_2) > f\,(x_1)\,.$$

This proves that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is strictly increasing.