# Double Integral

## MAT103 Calculus II

**Wang Fangyu, Indrakusuma Theresia Jasmine**

XIAMEN UNIVERSITY MALAYSIA
厦門大學馬來西亞分校

# Table of Contents

## Problem 1

*Find the volume of the solid that lies under the paraboloid z=x²+y², and above the region D in the X-Y plane bounded by the line y=-x and the parabola y=-x²+2.*

- Find the intersection of the line and parabola

$$-x = -x^2 + 2,$$
$$x_1 = 2, x_2 = -1$$

- So,the region D is

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 2, -x \leqslant y \leqslant -x^2 + 2\}$$

# Solution to Problem 1

## 1 Double Integral

- Use Fubini's Theorem to calculate the volume

$$
\begin{aligned}
V &= \int_{-1}^{2}\int_{-x}^{-x^2+2} \left(x^2+y^2\right)\, dydx \\
&= \int_{-1}^{2}\left[x^2y+\frac{1}{3}y^3\right]_{-x}^{-x^2+2} dy \\
&= \frac{1}{3}\int_{-1}^{2} -x^6 - x^4 + 4x^3 + 2x^2 + 8dy \\
&= \frac{1}{3}\left[-\frac{1}{7}x^7 - \frac{1}{5}x^5 + x^4 + \frac{2}{3}x^3 + 8x\right]_{-1}^{2} = \frac{233}{35}
\end{aligned}
$$

- So the volume of the solid is $\frac{233}{35}$.

### Problem 2

*(a) (7 points) Consider the expression below.*

$$\theta = \ln(e + \ln(e + \ln(e + \ln(e + \cdots))))$$

*Formulate the value $\theta$ as the limit of a sequence. Assuming the sequence converges, find the value of $\theta$ (up to 4th decimal place).*
*(b) (8 points) Show that the sequence is monontone, and hence it converges.*

# Solution to Problem 2(a)



- Consider the following sequence defined recursively:

$$a_0 = 0, \qquad a_{n+1} = \ln(e + a_n), \; n \geq 0.$$

- Then we have

$$\begin{aligned}
a_0 &= 0, \\
a_1 &= \ln(e + a_0) = \ln(e) = 1 \\
a_2 &= \ln(e + a_1) = \ln(e + \ln(e)) \\
a_3 &= \ln(e + a_2) = \ln(e + \ln(e + \ln(e))), \\
a_4 &= \ln(e + a_3) = \ln(e + \ln(e + \ln(e + \ln(e)))), \\
&\;\vdots
\end{aligned}$$

- We shall see $\theta$ arises as the limit of the process. If the sequence converges, we see that

$$\theta = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \ln(e + a_n) = \ln(e + \theta).$$

- That is, the limit solves the equation $\theta = \ln(e + \theta)$. Hence,

$$e^\theta = e + \theta.$$

- So the limit $\theta$ is the root of the equation, which is approximatley $\theta \approx 1.42037$.

- **Problem (8 points).** Show that the sequence is monontone, and hence it converges.
- Let $I = \{x : x \leq f(x)\}$ where $f(x) = \ln(e + x)$. Note that $0 \in I$. We check that if $y \in I$, then $f(y) \in I$.
- This is because $f$ is increasing, and so $y \leq f(y)$ implies

$$f(y) \leq f(f(y)).$$

- Now we have $a_0 \in I$. By induction it follows $a_n \in I$ for all $n$. Thus, $a_n \leq f(a_n) \leq a_{n+1}$. This shows $a_n$ is monotically increasing.

# Problem 3

1 Double Integral

### Problem 3

*Prove the Fubini's Theroem in the rectangle.*

- To prove the Fubini's Theroem in the rectangle, we firstly set a two-variables function which is continous on the given rectangle interval
- So we set a problem of: Let $f(x, y)$ is a continous function on $\mathbb{R}^3$, calculate the volume of the solid that lies under $f(x, y)$ and above the rectangle: $D = \{(x, y)\,|a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$.

# Solution of Problem 3

- Firstly, imagine y as an exact value,that is the green part of the following image.

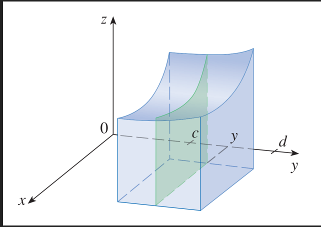


Figure 1: f(x,y) while y is a exact value

- So the formula $C = \int_a^b f(x, y)\, dx$ can be used to calculate the area of the green part of the picture

- Devided y into n equal subintervals

$$\Delta y = [y_0, y_1] = [y_1, y_2] = [y_2, y_3] = \dots = [y_{n-1}, y_n] \,.$$

- Let $y_i^* \subseteq [y_{i-1}, y_i]$, and for each subinterval, the volume is

$$V_i = \int_a^b f(x, y_i^*)\, dx.$$

- Let $g(y) = \int_a^b f(x, y)\, dx$

$$V_i = g(y_i^*) = \int_a^b f(x, y_i^*)\, dx.$$

- The Riemann summary of the volume of the solid is:

$$\sum_{i=1}^{n} g(y_i^*)\Delta y.$$

- n approach to infinity, so the Riemann summary is:

$$\lim_{n\to\infty}\sum_{i=1}^{n} g(y_i^*)\Delta y.$$

- By the defination of integral:

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n} g(y_i^*)\Delta y &= \int_c^d g(y)\,dy\\ &= \int_c^d\int_a^b f(x,y)\,dxdy = V.\end{aligned}$$

- Similarly, we could prove:

$$V = \int_a^b \int_c^d f(x, y)\, dydx.$$

- In conclusion:

$$V = \int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy.$$

- Thus, the Fubini's Theorem in the rectangle is proved.


THANK
YOU!!