Application of Integration: Retirement Annuity

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Section 1 Introduction

The Main Problems

- Fixed Contribution / Fixed Interest Rate
- Increasing Contribution / Fixed Interest Rate
- Increasing Contribution / Cyclical Interest Rate
- Periodic Contribution and Future Value Calculation
- Comparison of Annuity Balance and Future Value
- Complex Scenarios in Annuity Accumulation
- Special Annuity Calculations - The Perpetuity

Section 2 Basic Principles of Annuities

In retirement annuities, these functions model investment growth, particularly under continuous compounding.

Continuous Growth

Exponential functions epitomize continuous growth processes, essential in financial contexts like annuities. They are typically represented as

$$y(t) = Y(0)e^{kt}.$$

Parameters

- $Y(0)$ - initial value
- e - base of the natural logarithm
- k - growth rate
- t - time

This formula is vital for computing the future value of annuities, indicating the compound growth over time.

Future Value Calculation

The future value of an investment under continuous compounding is calculated using:

$$FV = PV \cdot e^{rt}$$

Components

- PV - present value or initial investment
- r - annual interest rate (decimal)
- t - time period in years

Inverse of Exponential Functions and Solving for Time and Rate

Inverse Relationship

Logarithmic functions, such as the natural logarithm (ln), are the inverses of exponential functions.

Example Calculation

For instance, to find the time required to double an investment:

$$t = \frac{\ln(FV/PV)}{r}$$

Section 3 Annuity Calculation in Different Situations

Use the fixed contribution and fixed interest rate as an ideal model to verify the feasibility of calculus calculations for annuities.

- Find the function consistent with annuity growth.
- Describe an annuity using the function.
- Use income stream to describe wages and functions to represent annuities.
- Use calculus ideas to convert functions into integrals and use integrals to calculate the annuity after 40 years under fixed contribution and fixed interest rates.
- Chart annuity account changes over time

The function of future value (FV) and present value (PV) can be expressed as:

$$FV = PV \times e^r t$$

- Contribute: 250\$ per month
- monthly interest rate: $\frac{r}{12}$
- Growth Rate: k = r = annual interest rate = 0.08(as a decimal)
- Future Value calculated over 40 years:
  - $$FV = \sum_{i=1}^{40\times 12} 250 \times e^{\frac{r}{12}\times(360-(i-1))}$$

The future value of these payment at time T would be approximately:

$$e^{r(T-t_i)}f(t_i)\Delta t$$

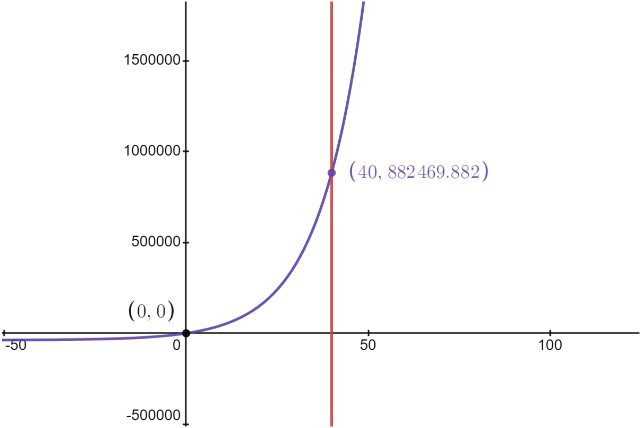
The all future values over the interval[0, T] would be:

$$FV \approx \sum_i e^{r(T-t_i)}f(t_i)\Delta t$$

As $\Delta t \to 0$, then we get formula for the FV of an annuity after 40 years ,compounded at interest rate r = 0.08 (as a decimal) continuously with payments at a rate of $f(t) = 250 \times 12$ dollars per year at time t:

$$\begin{aligned} FV &= \int_0^T e^{r(T-t)}f(t)\,dt \\ &= \int_0^{40} e^{0.08(40-t)}3000\,dt \\ &\approx 882,469.88 \end{aligned}$$

The graph shows the value of the amount with 0.08 as the interest rate , \$250 as the contribution per month changes over time.



Accumulated Value Calculation

- Employer matches for a 5% contribution rate, leading to a total annual contribution of 10% of the annual salary.
- Starting salary of \$30,000 with a 5% increase per year.
- Annual income given by $\$30,000 \times (1+0.05)^{t-1}$, accounting for yearly raises.
- Payment function $f(t) = 3000 \times 1.05^{t-1}$ dollars per year at time t.
- Future Value (FV) calculated over 40 years with continuous compounding at an 8% interest rate:

$$FV = \int_0^{40} e^{0.08\times(40-t)} \times 3000 \times 1.05^{t-1}\,dt$$

- Resulting Future Value: \$1,601,376.25.

Future Value with Increasing Contribution and Fixed Interest Rate

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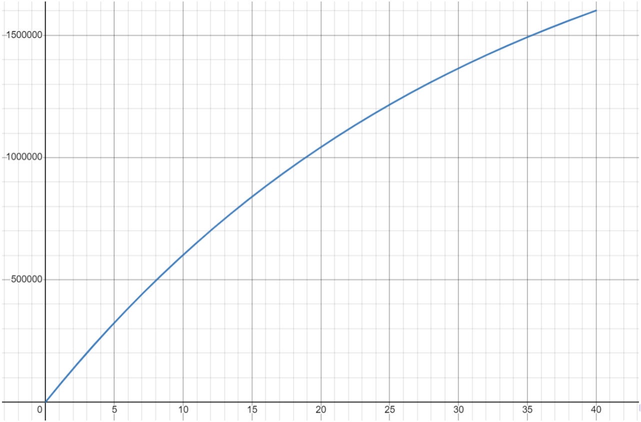
The graph shows the accumulated value over 40 years, with an initial contribution of \$3000 and a 5% annual increase, compounded with an 8% interest rate.



FV, r, T $\Rightarrow$ PV $\Rightarrow$ payment p.

- $PV = \sum_{i=1}^{n} FV_i \times e^{-rt_i}$
- $PV \approx \sum_{i} e^{-rt_i} g(t_i) \Delta t$
- $\Delta t \to 0,\ PV = \int_0^T e^{-rt} g(t)\, dt$
- $r = 6\%,\ T = 25,\ g(t) = 12p$
- Constant payment: $p = 10306.59301$

- In reality, interest rates are dynamic and subject to various influences, leading to fluctuations in the returns on investments.
- One can either make assumptions based on an average interest rate or attempt to model the actual pattern of returns.
- For the purposes of this project, let's consider an investment in an annuity tied to the prevailing prime interest rate.

- Suppose we consider a series of n payments that are separated by equal intervals of time and suppose that the payment amounts are $K_1, K_2, \ldots, K_n$.
- If the valuation rate of interest is i per period.
- $FV_i(t) = K_1(1+i)^{n-1} + K_2(1+i)^{n-2} + \cdots + K_{n-1}(1+i) + K_n$.

Scenario

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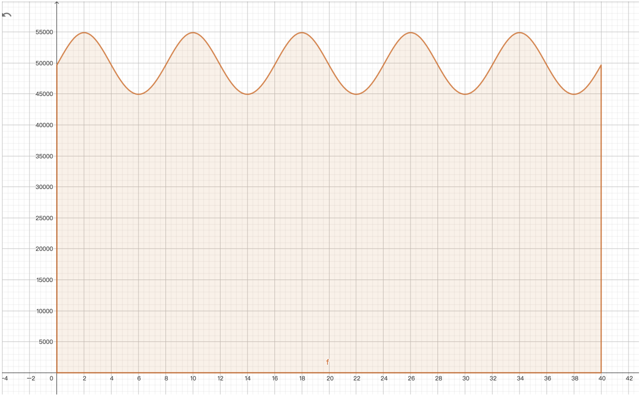
- The initial annual salary is $K = \$30,000$, you receive a consistent $r_a = 5\%$ salary increase annually, and you contribute 5% of your income to an annuity.
- This annuity features variable interest rates that continuously compound over the 40-year period, with corresponding employer contributions.
- We assume no annual effective inflation rate over the next 40 years.
- Yielded returns ranging from 6% to 10% annually.

- $r(t) = Asin(Bt + C) + D$
- The amplitude (A) is half the difference between the maximum and minimum values
- The frequency (B) is determined by the period of the cycle.
- The phase shift (C) is related to the current position in the cycle.
- The vertical shift (D) is the current interest rate.

- Cyclical Interest Rate: $r(t) = 2sin(\frac{\pi}{4}t) + 7$
- Calculation

$$
\begin{aligned}
FV &= \int_0^T e^{r(T-t)} f(t)\, dt \\
&= \int_0^{40} e^{0.05 \times [2\sin\left(\frac{\pi}{4}t\right)+7]} \times 30000 \times 0.05\, dt \\
&= \int_0^{40} e^{0.1\sin\left(\frac{\pi}{4}t\right)+3.5} \times 1500\, dt \\
&\approx 1.99268 \times 10^6
\end{aligned}
$$

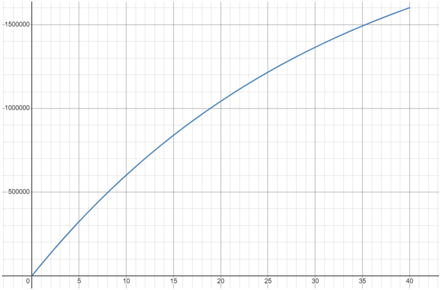
The graph shows the accumulated value which the interest rate exhibits a cyclic pattern, with the duration of each cycle, from the lowest to the highest point and vice versa, approximately spanning 8 years.



Set-up of Annuity Balance after Retirement

- Annuity balance $B(t)$ is given by $B(t) = b \cdot e^{rt} - \int_0^t w \cdot e^{r(t-s)}\, ds$.
- Parameters:
  - b: initial balance (\$1,601,376.25).
  - r: continuous annual interest rate (6%).
  - w: annual withdrawal rate (\$123,679.116176).
  - t: time in years (0 to 25).
  - s: variable of integration (time of each withdrawal).

Note: This setup visualizes the compounding effect on the annuity balance, accounting for regular withdrawals post-retirement.



Future value $FV(t)$

Annuity balance $B(t)$

Comparison Overview

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Annuity Balance vs. Future Value

Feature	**Annuity Balance** $B(t)$	**Future Value** $FV(t)$
Start Value	\$1,601,376.25	\$0
Growth Pattern	Decreasing due to withdrawals	Growth due to compounding
Interest Rate	6%	8%
Contributions	Annual withdrawals of \$123,679.1	Annual 10% salary contributed
Calculation Method	Present value of withdrawals	Future value of contributions
Expression	$B(t) = b \cdot e^{rt} - \int_0^t w \cdot e^{r(t-s)}\, ds$	$FV = \int_0^T e^{r(T-t)} f(t)\, dt$
Time Period	Retirement years 0 to 25	Working years 0 to 40
Graph Observation	Declines over time	Increases exponentially

Total Interest Calculation

Saving Phase

The total interest over 40 years can be calculated by the given salary formula and accumulated value without compound interest.

1. Salary formula: $f(t) = 3000 \times 1.05^{t-1}$.
2. Accumulated value without compound interest:

$$A_1 = \int_0^T f(t)\, dt = 326399.3$$

Total Interest Calculation

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2. Accumulated value without compound interest:

$$A_1 = \int_0^T f(t)\, dt = 326399.3$$

3. Total interest:

$$I_1 = 1,601,376.25 - 326,399.3 = 1,238,976.92$$

Total Interest Calculation

Withdrawal Phase

The total withdrawal over 25 years can be calculated by considering the constant withdrawal and accumulated value.

1. Constant withdrawal: $w = 123,679.12$.
2. Accumulated value:
$$A_2 = 25 \times w = 3,091,978$$
3. Total interest:
$$I_2 = A_2 - 1,601,376.25 = 1,490,601.75$$

Note: The present value of total withdrawals significantly exceeds the retirement balance, highlighting the importance of compound interest.

A perpetuity is a type of annuity that receives an infinite series of cash flows. It's an important concept in finance, especially when dealing with bonds and certain types of stocks. The basic formula for the present value of a perpetuity is given by:

$$PV = \frac{C}{r}$$

where PV is the present value of the perpetuity, C is the cash flow per period, and r is the discount rate.

To derive the relationship between the present value and cash flow of a perpetuity using integration, we consider the perpetuity as a continuous stream of cash flows. Assume that the cash flow is paid out in a continuous manner, with a continuous cash flow per year of C. The cash flow over an infinitesimally small time period dt is $C \cdot dt$. The present value of each small segment of cash flow is given by:

$$PV(dt) = \frac{C \cdot dt}{e^{rt}} \tag{1}$$

where e^{rt} is the discount factor for continuous compounding, and t is the time from now. (Wallstreetmojo Team, n.d.) Integrating this over an infinite time horizon, we get:

$$\begin{aligned} PV &= \int_0^\infty \frac{C \cdot dt}{e^{rt}} \\ &= C \int_0^\infty e^{-rt} dt \\ &= C \left[-\frac{1}{r} e^{-rt} \right]_0^\infty \\ &= \frac{C}{r} \end{aligned} \tag{2}$$

Calculation of Maximal Withdrawal

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Based on the above calculation, we aim to calculate the maximum cash flow that can be periodically withdrawn from a retirement account, Given the following conditions:

- The initial balance in the account at the time of retirement is \$1,601,376.25.
- The annual interest rate is fixed at 6% (i.e., $r = 0.06$).

Calculation of Maximal Withdrawal

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According to the formula for the present value of a perpetuity, the relationship between the annual interest rate and the account balance can be utilized to compute the maximum cash flow per period. The present value formula for a perpetuity is given by $PV = \frac{C}{r}$, where PV represents the present value of the perpetuity (in this case, the retirement account balance), C is the cash flow per period, and r is the annual interest rate. Rearranging this formula to solve for C yields:

$$\begin{aligned} C &= PV \times r \\ &= 1,601,376.25 \times 0.06 \\ &= 96,082.575 \end{aligned} \tag{3}$$

Therefore, at an annual interest rate of 6%, the maximum cash flow that can be withdrawn from this retirement account each year is $96,082.575. This implies that as long as the annual withdrawal amount does not exceed this figure, the principal in the account remains intact, theoretically allowing for an indefinite provision of such cash flows.

Section 4 Conclusion

Conclusion

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This study illustrates the correlation between integrals and retirement annuities, providing evidence for the fundamental concept and real-world applications.

The mathematical model posits that as Δt approaches 0 and **Present Value** PV becomes a function of t, the salary undergoes continuous changes. We segment the PV curve into smaller sections, computing rectangles to find the area under the PV function, representing **Future Value** FV.

However, in most scenarios, PV is not continuous; the salary in retirement annuities (PV) often remains constant over time. Despite this, our model is adaptable, allowing adjustments to variables. This adaptability underscores its significance and facilitates practical application.

In conclusion, retirement annuities can secure a reliable income stream, but careful consideration and professional advice are crucial in the decision-making process.

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Thank you